

63-3-3

Electrical Engineering Research Laboratory
The University of Texas
Austin, Texas

Report No. 129

February 28, 1963

**GENERAL DISPERSION RELATIONS FOR A
PARTIALLY-IONIZED GAS**

by

C. E. Prince, Jr.

F. X. Bostick, Jr.

Contract Nonr 375(14)
NR 371-032/3-27-61

OFFICE OF NAVAL RESEARCH
Washington, D. C.



403 127
403127
ASTIA
AS AD NO.

The research in this document has been made possible through the support and sponsorship extended by the Office of Naval Research under Contract Nonr 375(14). It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

Reproduction of this document in whole or in part is prohibited except with the permission of the office of origin. However, ASTIA is authorized to reproduce the document for "U. S. Governmental Purposes."

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government Agencies should be directed to

**ASTIA
Documents Service Center
Arlington Hall Station
Arlington 12, Virginia**

Department of Defense contractors must be established for ASTIA services or have their "need-to-know" certified by the cognizant military agency of their project or contract.

All other persons or organizations should apply to the

**U. S. Department of Commerce
Office of Technical Services
Washington 25, D. C.**

**ELECTRICAL ENGINEERING RESEARCH LABORATORY
THE UNIVERSITY OF TEXAS
Austin, Texas**

Report No. 129

February 28, 1963

**GENERAL DISPERSION RELATIONS FOR A
PARTIALLY-IONIZED GAS**

by

**C. E. Prince, Jr.
F. X. Bostick, Jr.**

**Contract Nonr 375(14)
NR 371-032/3-27-61**

**OFFICE OF NAVAL RESEARCH
Washington, D. C.**

TABLE OF CONTENTS

	Page
ABSTRACT	
I. INTRODUCTION	1
II. DERIVATION OF BASIC EQUATIONS	4
III. SPECIAL CASE SOLUTIONS	17
A. The MHD Approximations and a Simple Case	17
B. Additional Cases Factorable	21
C. The Fully-Ionized Case	23
D. The Partially-Ionized Case	31
IV. CONCLUSION	43
REFERENCES	

LIST OF FIGURES

- 1 Co-ordinate Axes
- 2 $\omega_e = \omega_{p_i}$
- 3 $\omega_i = \omega_{p_e}$
- 4 $\theta = 0^\circ$ Electromagnetic and Hydromagnetic Modes
- 5 $\theta = 0^\circ$ Acoustic Modes
- 6 Factored Mode
- 7 $\theta = 90^\circ$ Coupled Acoustic and Electromagnetic Modes

ABSTRACT

Beginning with Boltzmann's equation and the equation of momentum transfer for a partially-ionized gas, seven basic equations are derived for small-amplitude perturbations. These are used to obtain a determinant whose eigenvalues identify five modes of propagation in the most general case. A steady magnetic field and an arbitrary direction of propagation with respect to this field is supposed, and the complications introduced by this general approach are examined by a brief study of simpler cases.

It is shown that only by means of a high-speed electronic computer may attenuation factors and phase and group velocities be accurately determined. This paper surveys work done by other investigators, includes the approach taken by a research group at the Electrical Engineering Research Laboratory which has access to the Control Data Corporation 1604 Computer, and is only to be considered preliminary to later presentation of computer results.

I. INTRODUCTION

The dispersion relation is an equation which can be used to calculate phase and group velocities and attenuation and phase factors in a medium of propagation. It is well known that many parameters enter into this equation. The presence of a fixed magnetic field, a situation existing in the ionosphere, for example, greatly affects the form of the dispersion relation. This paper will outline the methods used by various investigators in an attempt to arrive at a general dispersion relation, without making the various approximations which usually make the equations more tractable.

Three types of waves have been identified in ionized gases - electromagnetic, hydromagnetic and electro-acoustic or space charge waves. Dispersion relations for the fully-ionized gas have been obtained under the special assumptions which make possible writing down neat, separate relations for each of the three types.^{1,2} However it has long been desired to find a relation which is most general and reduces to the simpler relations for each case.

We pose the problem as follows: Axes are set up in an ionized gas of three components - electrons, ions and neutral particles. We suppose a fixed magnetic field pointing in the positive z direction, $\vec{B} = \vec{k} B_0$. A plane wave front is supposed as shown, θ being the angle between the propagation direction and the \vec{B} vector. Consider all

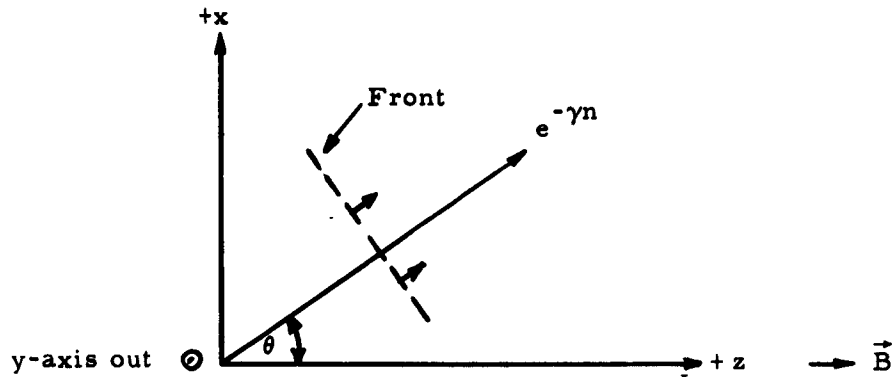


Figure 1 Co-ordinate Axes

derivatives with respect to the y-axis equal to zero. The quantity " η " in the $e^{-\gamma\eta}$ term is the unit distance along the propagation direction. A field quantity is then written

$$A = A_0 e^{-\gamma\eta} e^{j\omega t} = A_0 e^{-\gamma(x \sin \theta + z \cos \theta)} e^{j\omega t},$$

and hereafter $e^{j\omega t}$ is dropped whenever possible. The propagation constant is

$$\gamma = \alpha + j\beta$$

or, in general,

$$\gamma(\omega) = \alpha(\omega) + j\beta(\omega)$$

as γ is a function of ω alone (with angle θ and all other parameters fixed).

Of course, phase velocity and group velocity (in the " η " direction) are, respectively,

$$v_p = \frac{\omega}{\beta}$$

and

$$v_g = \frac{\partial \omega}{\partial \beta}.$$

A basic set of simultaneous, linear differential equations can be derived which applies to a mixture of three components and includes the effects of pressure and collisions. The eigenvalue solution of the determinant (of the homogeneous set) can be very complicated, so usually neutrals are disregarded. Further simplifications omit pressure gradients; further, collisions are ignored. As consideration of an arbitrary θ gets difficult, the directions $\theta = 0^\circ$ and 90° are chosen. Then, too, sometimes the displacement current term in Maxwell's equations is ignored, only low frequencies are considered, etc.

At least three considerations force these simplifications. One is the tremendous amount of algebra otherwise encountered. A second is the obvious correctness of the approximations in special cases. A third is the impossibility of exact calculation without a computer if the approximations aren't made.

This paper shows: (1) the derivation of the basic set of equations and (2) solutions of the equations in certain cases of increasing complexity. Lastly, there is a brief discussion of research requiring knowledge of plasma propagation phenomena.

II. DERIVATION OF BASIC EQUATIONS

We begin with the macroscopic approach and Boltzmann's equation

$$\frac{\partial d}{\partial t} + \vec{V} \cdot \nabla_{\vec{r}} d + \frac{\vec{F}}{m} \cdot \nabla_{\vec{v}} d = \left. \frac{\partial d}{\partial t} \right|_{\text{coll.}}$$

where "d" is the particle distribution function in phase space,

$d = f(x, y, z, V_x, V_y, V_z, t)$. Spitzer shows how the equation of momentum transfer and the equation of mass continuity are derived from this starting point. Chapman and Cowling show that for a gas of any number of components, an equation of momentum transfer can be written for each component, say for the j^{th} component,³

$$\begin{aligned} \rho_{m_j} \left[\frac{\partial \vec{V}_j}{\partial t} + (\vec{V}_j \cdot \nabla) \vec{V}_j \right] &= \rho_j (\vec{E} + \vec{V}_j \times \vec{B}) - \nabla \cdot \vec{\Omega}_j \\ &\quad - \rho_{m_j} \nabla \phi + \sum_{\substack{h \\ h \neq j}} \vec{P}_{jh} \end{aligned} \quad (1)$$

where

$$\rho_{m_j} = n_j m_j = \text{mass density}$$

$$\vec{V}_j = \text{avg. velocity at } \vec{R} \text{ of } j^{\text{th}} \text{ gas}$$

$$\rho_j = n_j Z_e = \text{charge density (} Z = 1 \text{ for singly-ionized gas)}$$

$$\vec{\Omega} = \text{stress tensor} = \nabla P_j \text{ in isotropic, inviscid case (} P_j = \text{kinetic gas pressure)}$$

$\nabla\phi$ = gradient of gravitational potential

\vec{P}_{jh} = momentum transferred to j^{th} particles by "h" particles.

n_j = particles/unit volume

m_j = mass/particle

e = electronic charge

Next, this equation is specialized to a mixture of ions, electrons and neutral particles where

1. $n_i = n_e = n$ = density $\left(\frac{\text{particles}}{3}\right)$ at a point (in a differential volume)

2. n_n = density of neutrals

3. $e = +q$ for ions, $-q$ for electrons, 0 for neutrals.

4. $\vec{P}_{jh} = -n_j n_h \beta_{jh} (\vec{V}_j - \vec{V}_h)$ where (see reference 4)

$$\beta_{jh} = \frac{f_{jh}}{n_h} \frac{m_j m_h}{m_j + m_h}$$

so that

$$\beta_{ie} = \frac{f_{ie}}{n} \frac{m_e m_i}{m_e + m_i} = f_{ie} \cdot \frac{m_e}{n},$$

and

$$\beta_{in} = \frac{f_{in}}{n_n} \frac{m_i m_n}{m_i + m_n} = \frac{f_{in}}{n_n} \frac{m_i}{2} = f'_{in} \frac{m_i}{n_n},$$

and

$$\beta_{en} = \frac{f_{en}}{n_n} \cdot \frac{m_e m_n}{m_e + m_n} = f_{en} \frac{m_e}{m_n},$$

if we assume m_i (ion mass) is equal to m_n (neutral mass), and

$$1 + \frac{m_e}{m_n} \approx \text{unity.}$$

(Here the "f's" are effective collision frequencies.)

For ion, electrons and neutrals, equation (1) becomes

Ions:

$$nm_i \frac{\partial \vec{V}_i}{\partial t} + nm_e f_{ie}' (\vec{V}_i - \vec{V}_e) + nm_i f_{in}' (\vec{V}_i - \vec{V}_n) = nq (\vec{E} + \vec{V}_i \times \vec{k} B_0) - \nabla p_i \quad (2)$$

Electrons:

$$nm_e \frac{\partial \vec{V}_e}{\partial t} + nm_e f_{ie}' (\vec{V}_e - \vec{V}_i) + nm_e f_{en}' (\vec{V}_e - \vec{V}_n) = -nq (\vec{E} + \vec{V}_e \times \vec{k} B_0) - \nabla p_e \quad (3)$$

Neutrals:

$$n_n m_n \frac{\partial \vec{V}_n}{\partial t} + nm_i f_{in}' (\vec{V}_n - \vec{V}_i) + nm_e f_{en}' (\vec{V}_n - \vec{V}_e) = -\nabla p_n \quad (4)$$

Hoping we do not invalidate grandiose ambitions, we have dropped the gravity and $(\vec{V} \cdot \nabla) \vec{V}$ terms, in order to avoid non-linearities and excessive algebraic complications, as is commonly done.

In these equations, it is necessary to note with great care that:

1. $\vec{E} = \vec{E}_0 + \delta \vec{E} = \delta \vec{E} = \text{steady-state field plus the small perturbation}$
 $(\vec{E}_0 \text{ is assumed zero, } \delta \vec{E} \text{ varying as } e^{j\omega t - \gamma \eta}).$

2. $\vec{B} = \vec{k} B_0 + \delta\vec{B}$ ($\vec{k} B_0 \gg \delta\vec{B}$, so $\delta\vec{B}$ is dropped from equations (2), (3) and (4).)

3. $n = N_{eo, io} + \delta n_{e, i}$ = undisturbed charge density plus perturbation
($N_{eo} = N_{io} = N_0$ hereafter, but $\delta n_e \neq \delta n_i$ necessarily).

4. $n_n = N_n + \delta n_n$ ($N_n \gg \delta n_n$; $N_n = N$ hereafter).

5. $P_{e, i} = P_{eo, io} + \delta p_{e, i} = P_0 + \delta p_{e, i}$

= steady state pressure plus perturbation

($P_0 = N_0 k T$, assuming $T_e = T_i = T_n$ and, also ∇P_0 or ∇P_n , i.e., space derivatives of unperturbed pressures, are zero in the homogeneous layers.)

6. $\vec{V}_{e, i, n} = \vec{V}_{eo, io, no} + \delta\vec{V}_{e, i, n} = \delta\vec{V}_{e, i, n}$ (The gases are at rest and we have dropped the "0" from equations (2), (3) and (4).)

7. $P_n = P_{no} + \delta p_n = P + \delta p_n$ with
($P = N_n k T$ and $\nabla P = 0$)

Continuing, we use equations (2), (3), (4) and the mass continuity equation for the j^{th} component,

$$\frac{\partial N_j}{\partial t} + N_j \nabla \cdot \vec{V}_j = 0 \quad (5)$$

along with the following equation of state for a perfect gas, derivable assuming a monomolecular gas with "1" degrees of freedom and adiabatic compression,

$$\delta n_j = \delta p_j \frac{N_{j0}}{P_{j0}} \left(\frac{l_j}{l_j + 2} \right) \quad (6)$$

the use of which with equation (5) can eliminate the perturbations in number density in favor of perturbations in pressure, for, substituting equation (6) in equation (5), we get

$$\frac{N_{j0}}{P_{j0}} \left(\frac{l_j}{l_j + 2} \right) \frac{\partial(\delta p_j)}{\partial t} + N_{j0} \nabla \cdot \vec{V}_j = 0$$

or

$$\frac{1}{P_{j0}} \left(\frac{l_j}{l_j + 2} \right) \frac{\partial p_j}{\partial t} + \nabla \cdot \vec{V}_j = 0 \quad (7)$$

dropping the δ notation.

We now proceed to derive a "generalized Ohm's Law" for the ternary mixture. The pattern is fairly familiar but certain steps need explicit presentation. Multiply equation (2) by $+q$ and divide by m_i ; multiply equation (3) by $-q$ and divide by m_e . Add the results to get

$$\begin{aligned} & \frac{\partial}{\partial t} \left[qn \vec{V}_i - qn \vec{V}_e \right] + qn f_{ie} \left(1 + \frac{m_e}{m_i} \right) (\vec{V}_i - \vec{V}_e) + qn \left[f'_{in} (\vec{V}_i - \vec{V}_n) - f_{en} (\vec{V}_e - \vec{V}_n) \right] \\ & = q^2 n \left[\frac{1}{m_i} + \frac{1}{m_e} \right] \vec{E} + q^2 n \left[\frac{\vec{V}_i}{m_i} + \frac{\vec{V}_e}{m_e} \right] \times \vec{k} B_0 + q \left[\frac{\nabla p_e}{m_e} - \frac{\nabla p_i}{m_i} \right] \end{aligned}$$

Letting $\vec{J} = qn (\vec{V}_i - \vec{V}_e) = N_0 q (\vec{V}_i - \vec{V}_e)$

$$\frac{1}{m_i} + \frac{1}{m_e} = \frac{1}{m_e}$$

$$1 + \frac{m_e}{m_i} = 1$$

and multiplying by $\frac{m_e}{N_o q^2}$,

we have,

$$\begin{aligned} & \frac{m_e}{N_o q^2} \left\{ \frac{\partial \vec{J}}{\partial t} + f_{ie} \vec{J} + q N_o \left[f'_{in} (\vec{V}_i - \vec{V}_n) - f_{en} (\vec{V}_e - \vec{V}_n) \right] \right\} \\ &= \vec{E} + m_e \left[\frac{m_e \vec{V}_i + m_i \vec{V}_e}{m_i m_e} \right] \times \vec{k} B_o + \frac{1}{N_o q} \left[\nabla p_e - \frac{m_e}{m_i} \nabla p_i \right] \quad (8) \end{aligned}$$

Examining the second term on the right, we have

$$\begin{aligned} & \frac{1}{m_i} \left[m_e (\vec{V}_i + \vec{V}_e - \vec{V}_e) + m_i (\vec{V}_e + \vec{V}_i - \vec{V}_i) \right] \times \vec{k} B_o \\ &= \frac{1}{m_i} \left[(\vec{V}_e - \vec{V}_i)(m_i - m_e) + m_i \vec{V}_i + m_e \vec{V}_e \right] \times \vec{k} B_o \\ &= \left[(\vec{V}_e - \vec{V}_i) + \frac{nm_i \vec{V}_i + n_e m_e \vec{V}_e}{n(m_i + m_e)} \right] \times \vec{k} B_o \quad \text{and,} \end{aligned}$$

defining

$$\vec{V}_p = \frac{n_i m_i \vec{V}_i + n_e m_e \vec{V}_e}{n(m_i + m_e)} \approx \frac{nm_i \vec{V}_i + nm_e \vec{V}_e}{nm_i},$$

we have this term equal to

$$\left[(\vec{V}_e - \vec{V}_i) + \vec{V}_p \right] \times \vec{k} B_o \text{ or } \left[-\frac{\vec{J}}{N_o q} + \vec{V}_p \right] \times \vec{k} B_o. \quad (9)$$

Now we examine the bracketed terms in the left-hand brace:

$$\begin{aligned} \left[f'_{in} (\vec{V}_i - \vec{V}_n) - f_{en} (\vec{V}_e - \vec{V}_n) \right] &= f'_{in} \left[\vec{V}_i - \vec{V}_n + \frac{m_e}{m_i} (\vec{V}_e - \vec{V}_e) + \frac{m_e}{m_i} (\vec{V}_i - \vec{V}_i) \right] \\ &\quad - f_{en} \left[\vec{V}_e - \vec{V}_n + \frac{m_e}{m_i} (\vec{V}_e - \vec{V}_e) + \vec{V}_i - \vec{V}_i \right] \\ &= f'_{in} \left[\left(1 - \frac{m_e}{m_i}\right) \vec{V}_i - \vec{V}_n + \frac{m_e}{m_i} (\vec{V}_e - \vec{V}_e + \vec{V}_i) \right] \\ &\quad - f_{en} \left[\left(1 - \frac{m_e}{m_i}\right) \vec{V}_e - \vec{V}_n + \frac{m_e}{m_i} (\vec{V}_e) + \vec{V}_i - \vec{V}_i \right] \end{aligned}$$

which becomes

$$\begin{aligned} (f_{en} - f'_{in}) \left[-\frac{m_e}{m_i} \vec{V}_e - \vec{V}_i + \vec{V}_n \right] &+ \left[f_{en} + \frac{m_e}{m_i} f'_{in} \right] (\vec{V}_i - \vec{V}_e) \\ &= - \left[f_{en} - f'_{in} \right] \left[\frac{m_e \vec{V}_e + m_i \vec{V}_i}{m_i} - \vec{V}_n \right] \\ &\quad + \left[f_{en} + \frac{m_e}{m_i} f'_{in} \right] (\vec{V}_i - \vec{V}_e) \end{aligned}$$

where, letting

$$(f_{en} - f'_{in}) = f_l,$$

we have

$$-f_l \left[\vec{V}_p - \vec{V}_n \right] + \left[f_{en} + \frac{m_e}{m_i} f'_{in} \right] (\vec{V}_i - \vec{V}_e). \quad (10)$$

We now substitute expressions (9) and (10) into equation (8), with

$$\left(f_{ie} + f_{en} + \frac{m_e}{m_i} f'_{in} \right) = f_3$$

to get

$$\frac{m_e}{q^2 N_o} \left[\frac{\partial \vec{J}}{\partial t} + f_3 \vec{J} \right] - \frac{m_e f_i}{q} (\vec{V}_p - \vec{V}_n) + \frac{\vec{J}}{N_o q} \times \vec{k} \vec{B}_o = \vec{E} + \vec{V}_p \times \vec{k} \vec{B}_o$$

and, since

$$\omega_e = \frac{q B_o}{m_e},$$

the cyclotron resonance frequency for electrons, we obtain finally

$$\begin{aligned} \frac{1}{\epsilon_o \omega_p^2} \left[\frac{\partial \vec{J}}{\partial t} + f_3 \vec{J} + \omega_e \vec{J} \times \vec{k} \right] &= \vec{E} + \vec{V}_p \times \vec{k} \vec{B}_o + \frac{B_o f_l}{\omega_e} (\vec{V}_p - \vec{V}_n) \\ &+ \frac{1}{N_o q} \left[\nabla p_e - \frac{m_e}{m_i} \nabla p_i \right]. \end{aligned} \quad (11)$$

Equation (11) is the general expression sought.^{5,6} We have introduced the electron plasma frequency

$$\omega_p^2 = \frac{N_o q^2}{m_e \epsilon_o}.$$

The equation is in the RMKS units. No approximations other than $1 \pm \frac{m_e}{m_i} = 1$ and $m_i = m_n$ have been made.

We now proceed to find the equation of motion for (1) the charged component and (2) the neutral component. Returning to equation (4) on page 6, we have

$$\rho_{m_n} \frac{\partial \vec{V}_n}{\partial t} + \left[\rho_{m_i} f'_{in} (\vec{V}_n - \vec{V}_i) + \rho_{m_e} (\vec{V}_n - \vec{V}_e) \right] = - \nabla p_n \quad (4')$$

and, examining the bracketed term as before, making similar approximations, we have

$$\begin{aligned} & - \rho_{m_i} f'_{in} \left[\vec{V}_i - \vec{V}_n + \frac{m_e}{m_i} (\vec{V}_i - \vec{V}_i) + \frac{m_e}{m_i} (\vec{V}_e - \vec{V}_e) \right] \\ & + \rho_{m_e} f_{en} (\vec{V}_n - \vec{V}_e + \vec{V}_i - \vec{V}_i + \frac{m_e}{m_i} (\vec{V}_e - \vec{V}_e)) \\ & = \rho_{m_e} [f_{en} - f'_{in}] (\vec{V}_i - \vec{V}_e) - \rho_{m_i} \left[f'_{in} + \frac{m_e}{m_i} f_{en} \right] \left[\vec{V}_i + \frac{m_e}{m_i} \vec{V}_e - \vec{V}_n \right] \end{aligned}$$

which may be written, recalling

$$f_{en} - f'_{in} = f'_l$$

$$N_o q (\vec{V}_i - \vec{V}_e) = \vec{J}$$

$$\vec{V}_p = \frac{m_i \vec{V}_i + m_e \vec{V}_e}{m_i} \cong \frac{\rho_{m_i} \vec{V}_i + \rho_{m_e} \vec{V}_e}{N_o m_i},$$

and defining

$$f_2 = \left(f'_{in} + \frac{m_e}{m_i} f_{en} \right),$$

as

$$\frac{m_e}{q} f_1 \vec{J} - N_o m_i f_2 (\vec{V}_p - \vec{V}_n)$$

which is also

$$\frac{B_o f_1}{\omega_e} \vec{J} + N_o m_i f_2 (\vec{V}_n - \vec{V}_p). \quad (12)$$

Substituting equation (12) into equation (4') we obtain

$$Nm_i \frac{\partial \vec{V}_n}{\partial t} = - \frac{B_o f_1}{\omega_e} \vec{J} - N_o m_i f_2 (\vec{V}_n - \vec{V}_p) - \nabla p_n \quad (13)$$

To obtain an equation of motion for the charged component of the gas, we add equations (2) and (3) to get

$$\begin{aligned} \rho_{m_e} \frac{\partial \vec{V}_e}{\partial t} + \rho_{m_i} \frac{\partial \vec{V}_i}{\partial t} + \rho_{m_i} f'_{in} (\vec{V}_i - \vec{V}_n) + \rho_{m_e} f_{en} (\vec{V}_e - \vec{V}_n) &= \vec{J} \times B_o \vec{k} \\ &- \nabla p_i - \nabla p_e \end{aligned}$$

or

$$\rho_{m_p} \frac{\partial \vec{V}_p}{\partial t} - \left[\rho_{m_i} f_{in} (\vec{V}_n - \vec{V}_i) + \rho_{m_e} f_{en} (\vec{V}_n - \vec{V}_e) \right] = \vec{J} \times \vec{k} B_0 - \nabla p_i - \nabla p_e$$

The bracketed terms have been shown equal to expression (12); so, on substitution, we have

$$N_o m_i \frac{\partial \vec{V}_p}{\partial t} = \frac{B_o f_l}{\omega_e} \vec{J} - N_o m_i f_2 (\vec{V}_p - \vec{V}_n) + \vec{J} \times \vec{k} B_o - \nabla p_p \quad (13')$$

Next, from the equation of mass continuity, in the form of equation (7), we can write for the electron gas

$$\frac{1}{p_o} \left(\frac{l}{l+2} \right) \frac{\partial (\delta p_e)}{\partial t} + \nabla \cdot \vec{V}_e = 0$$

This can be written as

$$\frac{1}{p_o} \left(\frac{l}{l+2} \right) \frac{\partial p_e}{\partial t} + \nabla \cdot \left(\vec{V}_p - \frac{\vec{J}}{N_o q} \right) = 0 \quad (14)$$

since

$$\begin{aligned} \left(\vec{V}_p - \frac{\vec{J}}{N_o q} \right) &= \frac{m_e \vec{V}_e + m_i \vec{V}_i}{m_i} - \frac{N_o q (\vec{V}_i - \vec{V}_e)}{N_o q} \\ &= \frac{m_e}{m_i} \vec{V}_e + \vec{V}_i - \vec{V}_i + \vec{V}_e \\ &= \left(1 + \frac{m_e}{m_i} \right) \vec{V}_e \approx \vec{V}_e \end{aligned}$$

For the charged gas, we have by adding equation (7) for the electron and ion gases together

$$\frac{1}{2P_o} \left(\frac{l}{l+2} \right) \frac{\partial p_p}{\partial t} + \nabla \cdot \left(\vec{V}_p - \frac{\vec{J}}{2N_o q} \right) = 0 \quad (15)$$

since

$$P_o = N_o k T_{e,i}$$

$$2P_o = (N_{eo} + N_{io}) k T_{e,i}$$

and

$$\begin{aligned} 2\vec{V}_p - \frac{\vec{J}}{N_o q} &= 2 \frac{m_e}{m_i} \vec{V}_e + 2 \vec{V}_i - \vec{V}_i + \vec{V}_e \\ &= \left(1 + 2 \frac{m_e}{m_i} \right) \vec{V}_e + \vec{V}_i \\ &= \vec{V}_e + \vec{V}_i \end{aligned}$$

For the neutral gas, we have

$$\frac{1}{P} \left(\frac{l_n}{l_n + 2} \right) \frac{\partial p_n}{\partial t} + \nabla \cdot \vec{V}_n = 0 \quad (16)$$

In conclusion, we write Maxwell's equations as

$$\nabla \times (\nabla \times \vec{E}) = -j\omega\mu \vec{J} + \omega^2 \mu \epsilon_o \vec{E} \quad (17)$$

and, collecting equations (11), (13), (13'), (14), (15), (16) and (17) together,

letting $m_i = m$, we have

Ohm's Law:

$$\frac{1}{\epsilon_o \omega_p^2} \left[(j\omega + f_3) \vec{J} + \omega_e \vec{J} \times \vec{k} \right] = \vec{E} + \vec{V}_p \times \vec{k} B_o + \frac{B_o f_1}{\omega_e} (\vec{V}_p - \vec{V}_n) + \frac{\nabla p_e}{N_o q} \quad (11)$$

Equation of Motion for the Charged Component:

$$j\omega N_o m \vec{V}_p = \frac{B_o f_1}{\omega_e} \vec{J} - N_o m f_2 (\vec{V}_p - \vec{V}_n) + \vec{J} \times \vec{k} B_o - \nabla p_p \quad (13')$$

Equation of Motion for the Neutral Component:

$$j\omega N m \vec{V}_n = - \frac{B_o f_1}{\omega_e} \vec{J} - N_o m f_2 (\vec{V}_n - \vec{V}_p) - \nabla p_n \quad (13)$$

Continuity Equation for the Electron Gas:

$$\frac{j\omega p_e}{P_o k} + \nabla \cdot \left(\vec{V}_p - \frac{\vec{J}}{N_o q} \right) = 0 \quad (14')$$

Continuity Equation for the Charged Gas:

$$\frac{j\omega p_p}{2P_o k} + \nabla \cdot \left(\vec{V}_p - \frac{\vec{J}}{2N_o q} \right) = 0 \quad (15)$$

Continuity Equation for the Neutral Gas:

$$\frac{j\omega p_n}{P k_n} + \nabla \cdot \vec{V}_n = 0 \quad (16)$$

Maxwell's Equations (Wave Equation):

$$\nabla \times (\nabla \times \vec{E}) = -j\omega\mu \vec{J} + \omega^2\mu\epsilon_0 \vec{E} \quad (17)$$

The statement of the last seven equations concludes this section.

Derivatives with respect to time have been taken. All p_j 's (pressures) are perturbations only; $\vec{k}B_0$ is the fixed field; define $\frac{1}{k} = \left(\frac{l}{l+2}\right)$ and $\frac{1}{k_n} = \left(\frac{l_n}{l_n+2}\right)$. We have supposed that $\nabla p_e \gg \frac{m_e}{m_i} \nabla p_i$. These seven equations have seven unknowns, which are written \vec{J} , \vec{V}_p , \vec{V}_n , p_e , p_i , p_n , \vec{E} . These seven equations are used in the next section. The reader would do well at this point to refer to Watanabe's thorough work on this analysis.^{5,6}

III. SPECIAL CASE SOLUTIONS

A. The MHD Approximations and a Simple Case:

As an illustration of the use of the basic seven equations and to show in a most simple way the method to be used later, we make what are called "magneto-hydrodynamic (MHD) approximations".^{1,2} We consider only a fully-ionized gas, with no neutrals, keep only one pressure term, set all collision frequencies equal to zero, disregard displacement current to get the following set:

$$\vec{E} = -\vec{V}_p \times \vec{k} B_o \quad (18)$$

$$j\omega N_o m \vec{V}_p = \vec{J} \times \vec{k} B_o - \nabla p_p \quad (19)$$

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu \vec{J} \quad (20)$$

$$\frac{j\omega}{2P_o k} p_p + \nabla \cdot \vec{V}_p = 0 \quad (21)$$

We first obtain from this set two equations in \vec{E} and \vec{V}_p by solving equation (21) for p_p and substituting into equations (19), and solving equation (20) for \vec{J} to substitute into eq. (19). We get

$$\vec{E} = -\vec{V}_p \times \vec{k} B_o \quad (22)$$

$$j\omega N_o m \vec{V}_p = -\frac{B_o}{j\omega\mu} \left[\nabla \times \nabla \times \vec{E} \right] \times \vec{k} + \frac{2P_o k}{j} \nabla(\nabla \cdot \vec{V}_p) \quad (23)$$

We now let

$$\vec{E} = \left[\vec{i} E_x + \vec{j} E_y + \vec{k} E_z \right] e^{-\gamma x \sin \theta - \gamma z \cos \theta}$$

and

$$\vec{V}_p = \left[\vec{i} V_x + \vec{j} V_y + \vec{k} V_z \right] e^{-\gamma x \sin \theta - \gamma z \cos \theta}$$

substituting into equations (22) and (23), separate each equation into \vec{i} , \vec{j} , \vec{k} components, and we get six equations:

$$E_x + B_o V_y = 0$$

$$E_y - B_o V_x = 0$$

$$E_z = 0$$

$$j\omega N_o m V_x - \frac{B_o}{j\omega\mu} \gamma^2 E_y - \frac{2P_o k}{j\omega} \left(\gamma^2 a_1^2 V_x - \frac{2P_o k}{j\omega} a_1 a_2 \gamma^2 \right) = 0$$

$$j\omega N_o m V_y + \frac{B_o}{j\omega\mu} \left[\gamma^2 a_2^2 E_x - \gamma^2 a_1 a_2 E_z \right] = 0$$

$$j\omega N_o m V_z - \frac{2P_o k}{j\omega} \left[\gamma^2 a_1 a_2 V_x + \gamma^2 a_2^2 V_z \right] = 0$$

where we have put $\sin \theta = a_1$ and $\cos \theta = a_2$, these not to be confused with attenuation factors.

We know that for a solution to exist the determinant of the coefficients must be identically zero. This is Δ_1 :

$$\Delta_1 = \begin{vmatrix} E_x & E_y & E_z & V_x & V_y & V_z \\ 1 & 0 & 0 & 0 & +B_o & 0 \\ 0 & 1 & 0 & -B_o & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{B_o \gamma^2}{j\omega\mu} & 0 & \left[j\omega N_o m - \frac{2P_o k}{j\omega} a_1^2 \gamma^2 \right] & 0 & -\frac{2P_o k}{j\omega} a_1 a_2 \gamma^2 \\ \frac{B_o}{j\omega\mu} a_2^2 \gamma^2 & 0 & -\frac{B_o}{j\omega\mu} a_1 a_2 \gamma^2 & 0 & j\omega N_o m & 0 \\ 0 & 0 & 0 & \frac{2P_o k}{j\omega} a_1 a_2 \gamma^2 & 0 & \left[\frac{2P_o k}{j\omega} a_2^2 \gamma^2 - j\omega N_o m \right] \end{vmatrix} \equiv 0$$

Most conveniently, an expansion leads to the following factors:

$$\left[j\omega N_o m - \frac{B_o^2}{j\omega\mu} (a_2 \gamma)^2 \right] \left[\frac{B_o^2 \gamma^2}{j\omega\mu} \left(\frac{2P_o k}{j\omega} a_2^2 \gamma^2 - j\omega N_o m \right) - j\omega N_o m \left(\frac{2P_o k}{j\omega} \gamma^2 - j\omega N_o m \right) \right] = 0$$

Thus we see three modes of propagation for an arbitrary angle, one mode for $\theta = 90^\circ$ and two for $\theta = 0^\circ$. For $\theta = 0^\circ$, $\alpha_1 = 0$ and $\alpha_2 = 1$, and we have

$$\begin{aligned}\gamma_1 &= j\omega \left[\frac{N_o m}{2P_o k} \right]^{\frac{1}{2}} & V_{ph_1} &= \left[\frac{2P_o k}{N_o m} \right]^{\frac{1}{2}} \\ \gamma_2 &= j\omega \left[\frac{N_o m}{B_o^2/\mu} \right]^{\frac{1}{2}} & V_{ph_2} &= \left[\frac{B_o^2/\mu}{N_o m} \right]^{\frac{1}{2}}\end{aligned}$$

where γ_1 represents a sound type or electro-acoustic wave, γ_2 the well-known Alfvén mode, where B_o^2/μ , the magnetic pressure, corresponds to $2P_o k$, the kinetic gas pressure.

We see that for the arbitrary angle, γ_2 and V_{ph_2} are found by simply substituting for B_o , $B_o \cos \theta$.

For $\theta = 90^\circ$, $\alpha_2 = 0$ and $\alpha_1 = 1$ and we have

$$\begin{aligned}\gamma_3 &= j\omega \left[\frac{N_o m}{2P_o k + B_o^2/\mu} \right]^{\frac{1}{2}} \\ V_{ph_3} &= \left[\frac{2P_o k + B_o^2/\mu}{N_o m} \right]^{\frac{1}{2}}\end{aligned}$$

and appropriate changes are made for an arbitrary angle. The kinetic and magnetic pressures both clearly affect the third mode.

In many physical situations, one of the terms B_o^2/μ or $2P_o k$ is much greater than the other and expressions simplify.

B. Additional Cases Factorable:

As long as the 6×6 determinant obtained from the \vec{E} and \vec{V}_p equation is factorable (and this sometimes requires some experimentation to find the right expansion), the γ 's for the various modes can be solved for without too much trouble. Suppose we have a set of equations thus:

$$\frac{j\omega}{\epsilon_o \omega_p^2} \vec{J} = \vec{E} + \vec{V}_p \times B_o \vec{k}$$

$$j\omega N_o m \vec{V}_p = \vec{J} \times \vec{k} B_o - \nabla p_p$$

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu \vec{J} + \omega^2 \mu \epsilon_o \vec{E}$$

$$\frac{j\omega}{2P_o k} p_p + \nabla \cdot \vec{V}_p = 0$$

Here we have included the displacement current and $\frac{\partial \vec{J}}{\partial t}$, but have ignored all other terms and collisions in the basic set of seven.

The two equations for \vec{E} and \vec{V}_p are:

$$\frac{j\omega}{\epsilon_o \omega_p^2} \left[-\frac{1}{j\omega\mu} \left(\nabla \times \nabla \times \vec{E} - \omega^2 \mu \epsilon_o \vec{E} \right) \right] = \vec{E} + \vec{V}_p \times \vec{k} B_o$$

and

$$j\omega N_o m \vec{V}_p = -\frac{B_o}{j\omega\mu} \left[\nabla \times \nabla \times \vec{E} - \omega^2 \mu \epsilon_o \vec{E} \right] \times \vec{k} + \nabla(\nabla \cdot \vec{V}_p) \frac{2P_o k}{j\omega}$$

In this case, after the 6×6 determinant is found and expanded we get:

1. For any angle θ : (c = velocity of light in vacuum)

$$\begin{aligned} & \left[\left(j\omega \epsilon_0 B_0^2 - \frac{B_0^2}{j\omega\mu} \gamma^2 \right) \left(\frac{2P_0 k}{j\omega} (a_2 \gamma)^2 - j\omega N_0 m \right) + j\omega N_0 m \left(\frac{2P_0 k}{j\omega} \gamma^2 - j\omega N_0 m \right) \right. \\ & \left. \left(1 - \frac{\omega^2}{\omega_p^2} - \frac{c^2}{\omega_p^2} \gamma^2 \right) \right] \cdot \left[\left(1 - \frac{\omega^2}{\omega_p^2} \right) \left[\frac{B_0^2}{j\omega\mu} (a_2 \gamma)^2 - j\omega N_0 m \left(1 - \frac{\omega^2}{\omega_p^2} - \frac{c^2}{\omega_p^2} \gamma^2 \right) \right] \right. \\ & \left. - j\omega \epsilon_0 B_0^2 \left[1 - \frac{\omega^2}{\omega_p^2} - \frac{c^2}{\omega_p^2} (a_1 \gamma)^2 \right] \right] = 0. \end{aligned} \quad (25)$$

which for $\theta = 0^\circ$ becomes

$$\left[\left(j\omega \epsilon_0 B_0^2 - \frac{B_0^2}{j\omega\mu} \gamma^2 \right) + j\omega N_0 m \left(1 - \frac{\omega^2}{\omega_p^2} - \frac{c^2}{\omega_p^2} \gamma^2 \right) \right]^2 \left[\frac{2P_0 k}{j\omega} \gamma^2 - j\omega N_0 m \right] = 0 \quad (26)$$

which suggests how the Alfvén mode is affected and shows again the common acoustic mode.

2. For $\theta = 90^\circ$, we have

$$\begin{aligned} & \left[\left(j\omega \epsilon_0 B_0^2 - \frac{B_0^2}{j\omega\mu} \gamma^2 \right) + \left(j\omega N_0 m - \frac{2P_0 k}{j\omega} \gamma^2 \right) \left(1 - \frac{\omega^2}{\omega_p^2} - \frac{c^2}{\omega_p^2} \gamma^2 \right) \right] \cdot \\ & \left[1 - \frac{\omega^2}{\omega_p^2} - \frac{c^2}{\omega_p^2} \gamma^2 \right] = 0. \end{aligned} \quad (27)$$

Explicit expressions for the γ 's have not been solved for as this and the next case are regarded only as "pattern-setters" for preparation on the more complex cases.

As the next case, suppose that the given set of equation is the following:

$$\frac{1}{\epsilon_0 \omega_p^2} \left[j\omega \vec{J} + \omega_e \vec{J} \times \vec{k} \right] = \vec{E} + \vec{V}_p \times \vec{k} B_0$$

$$j\omega N_0 m \vec{V}_p = \vec{J} \times B_0 \vec{k} - \nabla p_p$$

$$\nabla \times \nabla \times \vec{E} = -j\omega \mu \vec{J} + \omega^2 \mu \epsilon_0 \vec{E}$$

$$\frac{j\omega P_p}{2P_0 k} + \nabla \cdot \vec{V}_p = 0$$

This set has been solved in the same manner as before and it is found that the determinantal expansion does not factor. It has thus been tentatively concluded, and it is most probable, that no factors exist in the complex cases of physical interest. The next case treated moves us farther into this realm.

C. The Fully-Ionized Case

We now consider a fully-ionized plasma, and use the basic set of equations with collision terms set to zero (tenuous case). Our basic set then is:

$$\frac{1}{\epsilon_o \omega_p^2} [j\omega \vec{J} + \omega_e \vec{J} \times \vec{k}] = \vec{E} + \vec{V}_p \times \vec{k} B_o + \frac{\nabla p_e}{N_o q} \quad (28)$$

$$j\omega N_o m \vec{V}_p = \vec{J} \times \vec{k} B_o - \nabla p_p \quad (29)$$

$$\nabla \times \nabla \times \vec{E} = -j\omega \mu \vec{J} + \omega^2 \mu \epsilon_o \vec{E} \quad (30)$$

$$\frac{j\omega}{P_o k} p_e + \nabla \cdot \left(\vec{V}_p - \frac{\vec{J}}{N_o q} \right) = 0 \quad (31)$$

$$\frac{j\omega}{2 P_o k} p_p + \nabla \cdot \left(\vec{V}_p - \frac{\vec{J}}{2 N_o q} \right) = 0 \quad (32)$$

The two equations for \vec{E} and \vec{V}_p are:

$$\begin{aligned} & \left[1 - \frac{\omega_e^2}{\omega_p^2} \right] \vec{E} + \left[\frac{1}{\mu \epsilon_o \omega_p^2} \right] \nabla \times \nabla \times \vec{E} + \frac{1}{j\omega \mu} \left[\frac{\omega_e}{\epsilon_o \omega_p^2} \right] (\nabla \times \nabla \times \vec{E}) \times \vec{k} \\ & + j\omega \left[\frac{\omega_e}{\omega_p^2} \right] \vec{E} \times \vec{k} + \vec{V}_p \times \vec{k} B_o - \frac{1}{j\omega} \left[\frac{P_o k}{N_o q} \right] \nabla (\nabla \cdot \vec{V}_p) \\ & - \left[\frac{\epsilon_o P_o k}{N_o^2 q^2} \right] \nabla (\nabla \cdot \vec{E}) = 0. \end{aligned} \quad (33)$$

and

$$\begin{aligned}
& \left[\frac{B_o}{j\omega\mu} \right] (\nabla \times \nabla \times \vec{E}) \times \vec{k} + \left[j\omega\epsilon_o B_o \right] \vec{E} \times \vec{k} + j\omega N_o m \vec{V}_p \\
& - \frac{1}{j\omega} \left[2P_o k \right] \nabla (\nabla \cdot \vec{V}_p) - \left[\frac{\epsilon_o P_o k}{N_o q} \right] \nabla (\nabla \cdot \vec{E}) = 0. \quad (34)
\end{aligned}$$

In the same way as previously, we put in the exponential forms for \vec{E} and \vec{V}_p and find the six homogeneous, simultaneous, linear equations. Next the 6×6 determinant of the co-efficients is set equal to zero. This determinant is shown on page 26 (Δ_2).

The solution of the eigenvalue problem is an interesting exercise. First, for $\theta = 0^\circ$, the determinant is factorable into two quadratic equations in γ^2 and the four modes are as follows:

$$\begin{aligned}
\gamma_{1,2}^2 &= -\omega^2 \frac{N_o m}{2P_o k} + \omega^2 \frac{N_o m}{P_o k} \left[\omega_{p_i}^2 \left(\frac{1}{\omega^2} - \frac{1}{\omega_{p_e}^2} \right) \right. \\
&\quad \left. \pm \omega_{p_i}^2 \sqrt{\left(\frac{1}{\omega^2} - \frac{1}{\omega_{p_e}^2} - \frac{1}{2\omega_{p_i}^2} \right)^2 + \frac{1}{\omega_{p_i}^2} \left(\frac{1}{\omega^2} - \frac{1}{\omega_{p_e}^2} \right)} \right]
\end{aligned}$$

and

$$\begin{aligned}
\gamma_3^2 &= \frac{1 + \frac{B_o^2}{c^2 \mu N_o m} - \frac{\omega^2}{\omega_{p_e}^2} \left(1 - \frac{\omega_e}{\omega} \right)}{-\frac{B_o^2}{\omega^2 \mu N_o m} + \frac{c^2}{\omega_{p_e}^2} \left(1 - \frac{\omega_e}{\omega} \right)}
\end{aligned}$$

$$\begin{array}{ccccc}
 E_x & E_y & E_z & V_x & V_y & V_z \\
 \left[\begin{array}{l}
 A - \gamma^2 (B a_2^2 - F a_1^2) \\
 -(D - C a_2^2 \gamma^2) \\
 \gamma^2 a_1 a_2 [B + F] \\
 -(K - J a_2^2 \gamma^2) \\
 \gamma^2 a_1 a_2 N \\
 \gamma^2 a_1^2 N
 \end{array} \right] &
 \left[\begin{array}{l}
 D - C \gamma^2 \\
 A - B \gamma^2 \\
 0 \\
 0 \\
 0 \\
 (K - J \gamma^2)
 \end{array} \right] &
 \left[\begin{array}{l}
 \gamma^2 a_1 a_2 [B + F] \\
 -\gamma^2 a_1 a_2 C \\
 A - \gamma^2 (B a_1^2 - F a_2^2) \\
 -\gamma^2 a_1 a_2 J \\
 \gamma^2 a_2^2 N \\
 \gamma^2 a_1 a_2 N
 \end{array} \right] &
 \left[\begin{array}{l}
 \gamma^2 a_1^2 E \\
 -B_0 \\
 E \gamma^2 a_1 a_2 \\
 0 \\
 \gamma^2 a_1 a_2 M \\
 L + M \gamma^2 a_1^2
 \end{array} \right] &
 \left[\begin{array}{l}
 B_0 \\
 0 \\
 0 \\
 L \\
 0 \\
 0
 \end{array} \right] &
 \left[\begin{array}{l}
 \gamma^2 a_1 a_2 E \\
 0 \\
 \gamma^2 a_2^2 E \\
 0 \\
 L + M \gamma^2 a_2^2 \\
 \gamma^2 a_1 a_2 M
 \end{array} \right] \\
 & & & & & = 0
 \end{array}$$

where $A = \left[1 - \frac{\omega^2}{\omega_p^2} \right] \quad E = -\frac{1}{j\omega} \left[\frac{P_0 k}{N_0 q} \right] \quad K = [j\omega \epsilon_0 B_0]$

$$B = \left[\frac{1}{\mu \epsilon_0 \omega_p^2} \right] \quad F = -\left[\frac{\epsilon_0 P_0 k}{N_0 q^2} \right] \quad L = j\omega N_0 m_i$$

$$C = \frac{1}{j\omega u} \left[\frac{\omega_e}{\epsilon_0 \omega_p^2} \right] \quad J = \frac{1}{j\omega u} [B_0] \quad M = -\frac{1}{j\omega} [2P_0 k]$$

$$D = j\omega \left[\frac{\omega_e}{\omega_p^2} \right] \quad N = -\left[\frac{\epsilon_0 P_0 k}{N_0 q} \right]$$

Δ_2

and

$$\gamma_4^2 = \frac{1 + \frac{B_o^2}{c^2 \mu N_o m} - \frac{\omega^2}{\omega_{pe}^2} \left(1 + \frac{\omega_e}{\omega}\right)}{-\frac{B_o^2}{\omega^2 \mu N_o m} + \frac{c^2}{2 p_e} \left(1 + \frac{\omega_e}{\omega}\right)}$$

It will be noticed that $\gamma_{1,2}$ is related to the original acoustic wave but is modified, and $\gamma_{3,4}$ is related to the original Alfvén modes.

As a check on these results, let us suppose that $B_o = 0$. Then

$\gamma_3^2 = \gamma_4^2$ and becomes

$$\gamma_{3,4}^2 = \frac{1 - \frac{\omega^2}{\omega_{pe}^2}}{\frac{c^2}{\omega_{pe}^2}} = - \frac{\omega^2 \left[1 - \frac{\omega_{pe}^2}{\omega^2}\right]}{c^2}.$$

Therefore

$$\gamma_{3,4} = \frac{j\omega \left[1 - \frac{\omega_{pe}^2}{\omega^2}\right]^{\frac{1}{2}}}{c}$$

and

$$V_{ph_{3,4}} = \frac{c}{\left[1 - \frac{\omega_{pe}^2}{\omega^2}\right]^{\frac{1}{2}}}$$

This is a familiar result.^{1,2}

Suppose further that the ratios $\frac{V_a}{c}$ and $\frac{V_a}{\omega}$ are small, where V_a is the simple Alfvén velocity, and we obtain

$$\begin{aligned} \gamma_{3,4}^2 &= \frac{1 - \frac{\omega^2}{\omega_{pe}^2} \left(1 \mp \frac{\omega_e}{\omega}\right)}{\frac{c^2}{\omega_{pe}^2} \left(1 \mp \frac{\omega_e}{\omega}\right)} \\ &= \frac{\omega^2 \left[1 - \frac{\omega_{pe}^2}{\omega^2} \frac{1}{\left(1 \mp \frac{\omega_e}{\omega}\right)}\right]}{c^2} \end{aligned}$$

so that

$$V_{ph_3} = \frac{c}{\left[1 - \frac{\omega_{pe}^2}{\omega^2} \frac{1}{\left(1 - \frac{\omega_e}{\omega}\right)}\right]^{\frac{1}{2}}}$$

and

$$V_{ph_4} = \frac{c}{\left[1 - \frac{\omega_{pe}^2}{\omega^2} \frac{1}{\left(1 + \frac{\omega_e}{\omega}\right)}\right]^{\frac{1}{2}}}$$

These two results are also familiar.^{1,2} V_{ph_3} is the phase velocity of the "extraordinary" wave and V_{ph_4} that of the "ordinary" wave.

Figures 2 and 3 show the variation in γ_3 and γ_4 with ω , for the complete expressions on page 28. Of course,

$$\omega_i = \frac{q B_o}{m_i}$$

and

$$\omega_{p_i}^2 = \frac{N_o q^2}{m_i \epsilon_o}$$

When γ^2 is positive, the wave does not propagate, but is evanescent. The γ_4 mode is essentially connected with the ions and the γ_3 mode with the electrons.

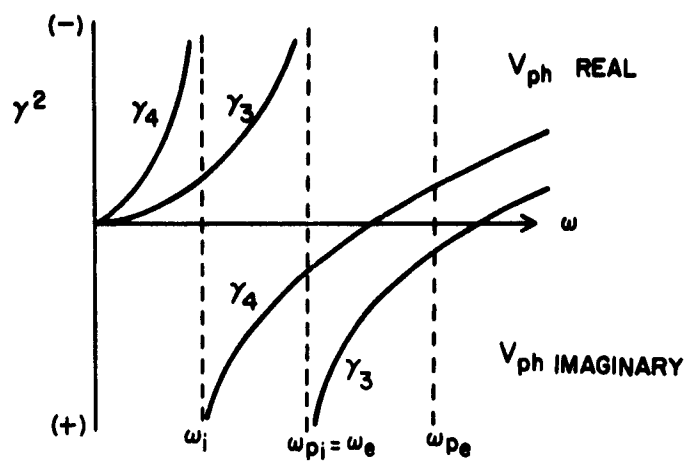
We shall not consider $\gamma_{1,2}$ of the acoustic modes. We refer the interested reader to Pai's paper.⁷

Next, we may examine the case $\theta = 90^\circ$. When we let $\alpha_2 = 0$ and $\alpha_1 = 1$ in Δ_2 and solve, we find one mode immediately:

$$\gamma_1^2 = \frac{\omega_{p_e}^2}{c^2} \left[1 - \frac{\omega^2}{\omega_{p_e}^2} \right] = \frac{1 - \frac{\omega^2}{\omega_{p_e}^2}}{\frac{c^2}{\omega_{p_e}^2}}$$

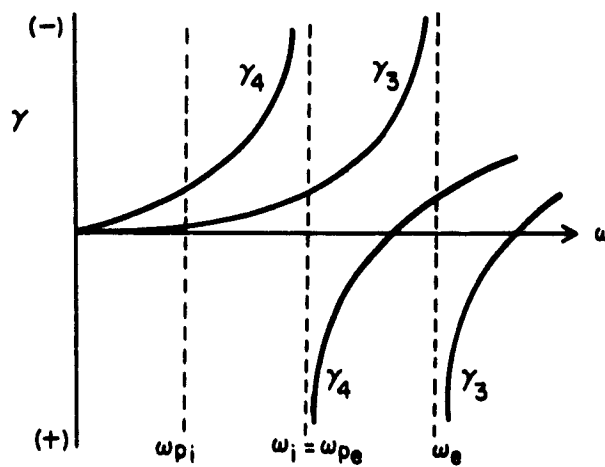
which is the electromagnetic mode found before the $\theta = 0^\circ$ case, page 27.

However, this simple factor is the only one possible and we are left with a cubic equation in γ^2 .



$$\omega_e = \omega_{pi}$$

FIG. 2.



$$\omega_i = \omega_{pe}$$

FIG. 3.

This case as well as the $\theta = 0^\circ$ case is discussed extensively by Tanenbaum.⁸ However, the case for arbitrary θ is a very difficult problem and has not been solved, and it appears to be a computer problem for special cases.

This problem has up to now disregarded electron and ion collisions. Usually this is taken as considering infinite conductivity. However, if f_{ie} is included in the basic set, equation (28) on page 24 is written as

$$\frac{1}{\epsilon_0 \omega_p^2} \left[(j\omega + f_3) \vec{J} + \omega_e \vec{J} \times \vec{k} \right] = \vec{E} + \vec{V}_p \times \vec{k} B_0 + \frac{\nabla p_e}{N_0 q}$$

where, instead of f_3 being $(f_{ie} + f_{en} + \frac{m_e}{m_i} f_{in})$ as in the basic set, it is only f_{ie} . This insertion changes the 6×6 determinant Δ_2 very little, but enough to introduce complex γ 's so that attenuation of the modes is caused. It is known that at very - low frequencies, MHD effects hold true and at very high frequencies, the Appleton-Hartree equations can now be derived.¹⁶ Computer calculation of attenuation and phase factors for cases of special interest is possible, as before.

D. The Partially-Ionized Case:

In the ionosphere and in most man-made plasmas, it cannot be assumed that full-ionization exists. To study this case, one must employ each of the basic seven equations on pages 16 and 17. We use the same

method as before, i.e., solve for two equations in \vec{E} and \vec{V}_p , and find the eigenvalues of the 6×6 determinant obtained after substitution of the exponential forms.

This is a long and demanding project with most excellent chances for algebraic errors. We will present the two equations in \vec{E} and \vec{V}_p , which become equations (33) and (34) if N , the neutral number density, is set to zero and all collision terms - f_1 , f_2 , f_3 - are also zero, (as would be expected if the equations are correct). These equations for the partially-ionized case, including the effects of collision and pressure terms, are on pages 33-34. The 6×6 determinant is on page 35 (Δ_3). Three independent checks have been made of these very complex expressions.

Once again we are presented with the problem of solution for $\theta = 0^\circ$, $\theta = 90^\circ$, and arbitrary θ . First, it can be seen, after much algebra, that for $\theta = 0^\circ$, the determinant factors into two polynomials, one a quadratic in γ^2 and, another, a cubic in γ^2 . There are thus five modes of propagation.

Secondly, for $\theta = 90^\circ$, the factors are a first degree γ^2 polynomial and a quartic in γ^2 (and a very complicated one). Again we have five modes.

Tanenbaum has done an excellent analysis of these two special cases and has shown qualitatively some very interesting results.¹⁰ For $\theta = 0^\circ$; the variation of the roots of the quadratic are as shown in Fig. 4.

$$\begin{aligned}
& \underbrace{\left[\frac{B_o^2 f_1^2}{\omega_e^2 N_o m f_2} - \frac{(j\omega + f_3)}{\epsilon_o \omega_p^2} \right]}_C \nabla \times \nabla \times \vec{E} + \underbrace{\left[\frac{B_o^2 f_1}{\omega_e N_o m f_2} - \frac{\omega_e}{\epsilon_o \omega_p^2} \right]}_D (\nabla \times \nabla \times \vec{E}) \times \vec{k} \\
& \underbrace{\left[\frac{\omega_p^2}{\omega_p^2} - 1 + j\omega \left(\frac{B_o^2 f_1^2 \epsilon_o}{\omega_e^2 N_o m f_2} - \frac{f_3}{\omega_p^2} \right) \right]}_A \vec{E} + \underbrace{\left[\frac{\epsilon_o P_o k}{N_o^2 q^2} \left[1 - \frac{B_o f_1 q}{\omega_e m f_2} \right] \right]}_G \nabla (\nabla \cdot \vec{E}) \\
& \underbrace{\left[\frac{B_o^2 f_1}{\omega_e N_o m f_2} - \frac{\omega_e}{\epsilon_o \omega_p^2} \right]}_B \vec{E} \times \vec{k} - \underbrace{\left[\frac{B_o f_1}{\omega_e f_2} \right]}_E \vec{V}_p \\
& + j\omega \epsilon_o \left[\frac{B_o^2 f_1}{\omega_e N_o m f_2} - \frac{\omega_e}{\epsilon_o \omega_p^2} \right] \vec{E} \times \vec{k} - \vec{V}_p \times k B_o + j\omega \left[\frac{B_o f_1}{\omega_e f_2} \right] \vec{V}_p \\
& \underbrace{\left[\frac{P_o k}{j\omega N_o q} \left[1 - \frac{2B_o f_1 q}{\omega_e m f_2} \right] \right]}_F \nabla (\nabla \cdot \vec{V}_p) = 0
\end{aligned}$$

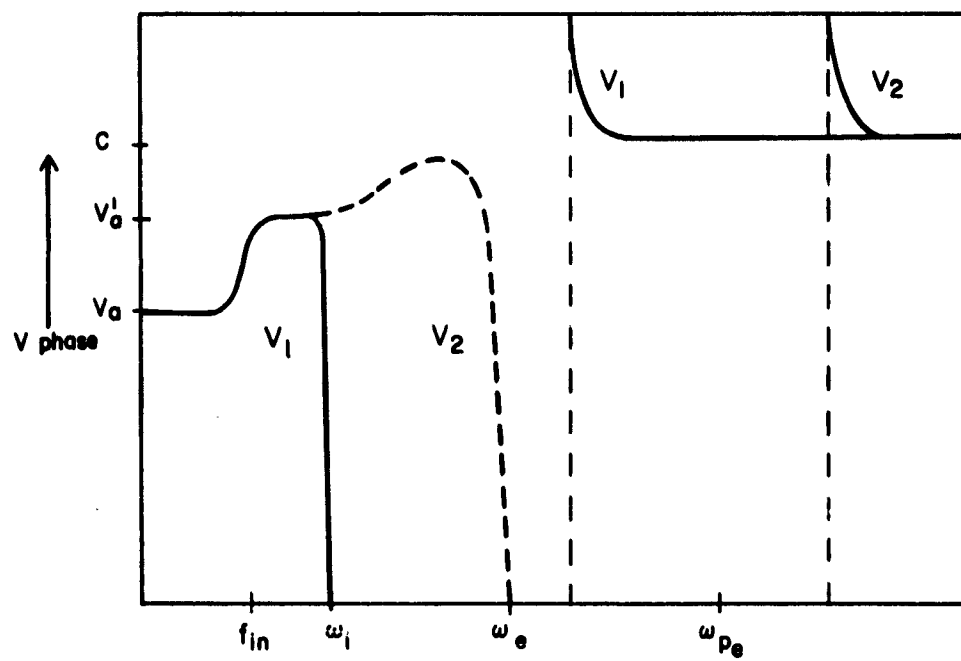
(A)

$$\begin{aligned}
& \underbrace{\left[\frac{B_0 f_1}{\omega_e f_2} \right] \left[\frac{N}{N_0 \mu} \right]}_J + \underbrace{\frac{B_0}{j\omega \mu} \left[\frac{j\omega N + N_0 f_2}{N_0 f_2} \right]}_K \left[\nabla \times (\nabla \times \vec{E}) \right] \times \vec{k} + \underbrace{\left[\frac{-B_0 f_1}{\omega_e f_2} \right] \left[\frac{N_0^2 \epsilon_0}{N_0} \right]}_H \vec{E} \\
& + j\omega \epsilon_0 B_0 \underbrace{\left[\frac{j\omega N + N_0 f_2}{N_0 f_2} \right] \vec{E} \times \vec{k}}_I - \underbrace{\left[\frac{B_0 \epsilon_0 P k_\eta}{N_0 m f_2} \right] \nabla (\nabla \cdot (\vec{E} \times \vec{k}))}_P + \underbrace{\frac{1}{j\omega} \left[\frac{\epsilon_0 P_0 k P k_\eta}{N_0^2 m f_2 q} \right]}_S \nabla (\nabla \cdot \nabla (\nabla \cdot \vec{E})) \\
& - \underbrace{\left[\frac{\epsilon_0 P_0 k}{N_0 q} \left(\frac{j\omega N + N_0 f_2}{N_0 f_2} \right) + \frac{B_0 f_1}{\omega_e} \left(\frac{\epsilon_0 P k}{N_0 m f_2} \right) \right] \nabla (\nabla \cdot \vec{E}) + \left[j\omega m (N + N_0) - \frac{\omega^2 N m}{f_2} \right]}_N \vec{V}_p \\
& - \underbrace{\left[\left(\frac{2 P_0 k N + P k_\eta N_0}{N_0 f_2} \right) - j \left(\frac{2 P_0 k + P k_\eta}{\omega} \right) \right] \nabla (\nabla \cdot \vec{V}_p)}_M \\
& - \underbrace{\left[\frac{2 P_0 k P k_\eta}{\omega^2 N_0 m f_2} \right] \nabla [\nabla \cdot (\nabla \cdot \vec{V}_p)] + \left[\frac{P k_\eta B_0}{\omega^2 N_0 m f_2 \mu} \right] \nabla \{ \nabla \cdot [(\nabla \times (\nabla \times \vec{E})) \times \vec{k}] \}}_R = 0
\end{aligned}$$

(B)

$$\begin{array}{ccccc}
E_x & E_y & E_z & V_x & V_y & V_z \\
\hline
[A - \gamma^2(Ca_2^2 - Ga_1^2)] & [B - D\gamma^2] & [\gamma^2 a_1 a_2 (C + G)] & [E + F\gamma_1^2 a_1^2] & [-B_0] & [\gamma^2 a_1 a_2 F] \\
[D\gamma^2 a_2^2 - B] & [A - C\gamma^2] & [-\gamma^2 a_1 a_2 D] & [B_0] & [E] & 0 \\
[\gamma^2 a_1 a_2 (C + G)] & 0 & [A - \gamma^2 (Ca_1^2 - Ga_2^2)] & [\gamma^2 a_1 a_2 F] & 0 & [E + F\gamma^2 a_2^2] \\
[K\gamma^2 a_2^2 - I] & [H - J\gamma^2] & [-\gamma^2 a_1 a_2 K] & 0 & [L] & 0 \\
[\gamma^2 a_1 a_2 (S\gamma^2 + J - N)] & [-\gamma^2 a_1 a_2 (Q\gamma^2 + P)] & [\gamma^2 a_2^2 (S\gamma^2 - N) + H - J\gamma^2 a_1^2] & [-\gamma^2 a_1 a_2 (M + R\gamma^2)] & 0 & [L - \gamma^2 a_2^2 (M + R\gamma^2)] \\
[H - \gamma^2 a_2^2 J + \gamma^2 a_1^2 (S\gamma^2 - N)] & [I - K\gamma^2 - \gamma^2 a_1^2 (P + Q\gamma^2)] & [\gamma^2 a_1 a_2 (J + S\gamma^2 - N)] & [L - \gamma^2 a_1^2 (M + R\gamma^2)] & 0 & [-\gamma^2 a_1 a_2 (M + R\gamma^2)]
\end{array}$$

= 0



$$\theta = 0^\circ$$

ELECTROMAGNETIC AND HYDROMAGNETIC MODES

FIG. 4.

The following statements apply to Fig. 4:

1. V_1 is the phase velocity of the usual left-circularly-polarized wave associated with the ions, V_2 that of the right-circularly-polarized wave associated with the electrons. Resonances occur at the cyclotron frequencies.

2. For very low frequencies (MHD), the only propagating mode is the Alfvén mode associated with the entire gas, and

$$V_a = \left[\frac{\frac{B_o^2}{\mu}}{\text{Total Mass Density}} \right]^{\frac{1}{2}}$$

3. As the frequency reaches f_{in} , the neutrals cease to take part and we have an Alfvén mode in the charged gas only

$$V_a' = \left[\frac{\frac{B_o^2}{\mu}}{\text{Charged Mass Density}} \right]^{\frac{1}{2}}$$

4. As the frequency continues to increase, we have the Alfvén modes ceasing to be effective and we pass to the magneto-ionic limits as in Figs. 2 and 3. The dispersion relation becomes approximately of Appleton-Hartree form.

5. These curves are drawn without considering appreciable collisions, but when collisional damping effects are effective, the waves can propagate with attenuation at all frequencies. For high values of f_1, f_2, f_3 the resonant effects are greatly diminished.

Next, for the cubic in γ^2 , the roots take the form shown in

Fig. 5.

The following statements apply to the curves:

1. The symbols U stand for the velocity of sound in the indicated gas.
2. At very-low frequencies, only the acoustic wave in the total gas is propagated,

$$U_a = \left[\frac{k_n P + 2 k P_o}{\text{Total Mass Density}} \right]^{\frac{1}{2}}$$

3. When f'_{in} is reached, coupling to the neutrals decreases, and we have an acoustic wave in the charged gas

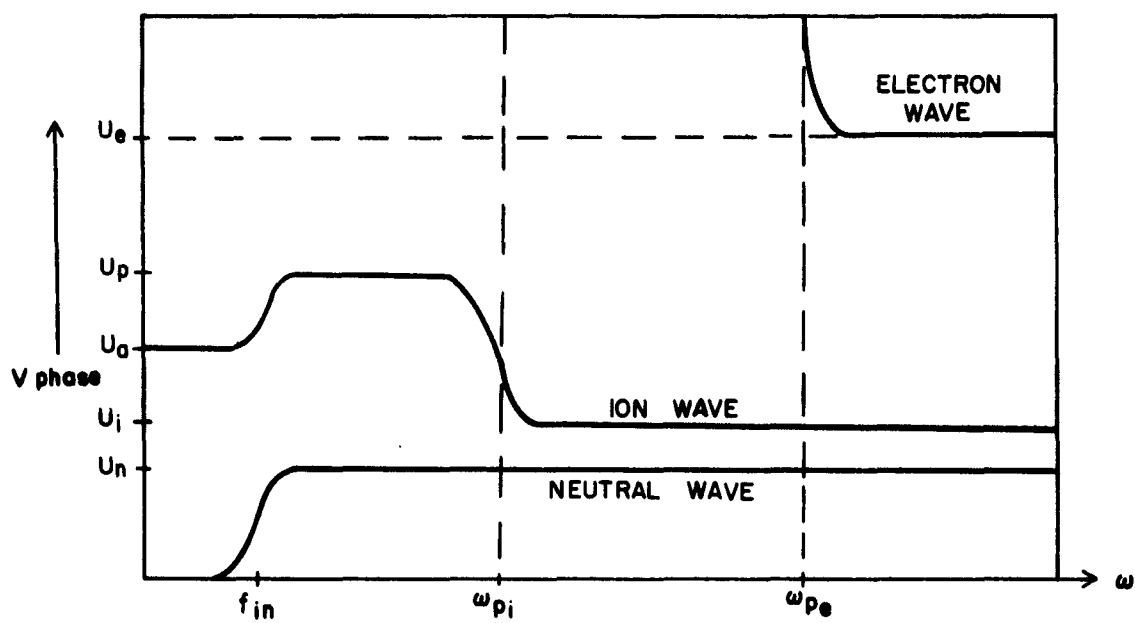
$$U_p = \left[\frac{2 k P_o}{\text{Charged Density}} \right]^{\frac{1}{2}}$$

and another in the neutral gas,

$$U_n = \left[\frac{k_n P}{N m} \right]^{\frac{1}{2}}$$

4. As ω_{p_i} is reached, we see that coupling between the ions and electrons decreases, so we have simply U_i .

5. The electron oscillations are greatly attenuated unless $\omega > \omega_{p_e}$; the phase velocity can be very high for $\omega \approx \omega_{p_e}$, but decreases to U_e as $\omega \rightarrow \infty$.



$$\theta = 0^\circ$$

ACOUSTIC MODES

FIG. 5.

6. Tanenbaum states some of the high-frequency results here cannot be trusted as they might pertain to wave-lengths (for high enough ω 's) which are shorter than the Debye distance.¹⁰

For the case of $\theta = 90^\circ$, one root, as has been said, can be quickly obtained, just as in the fully-ionized case on page 29. The factor is

$$\gamma^2 = \frac{[LA - EH]}{[LC - JE]}$$

with L, A, G, J, E drawn from the definitions in equations (B) and (A), page 29. This mode is obviously related to the fully-ionized mode. Tanenbaum affirms this by showing Figure 6.

All other waves propagating at $\theta = 90^\circ$ are coupled longitudinal and transverse waves (the E vector is at an acute angle with propagation direction). This fourth order polynomial in γ^2 has not been solved in general. However, by the use of known limits and assuming continuity of the roots, Tanenbaum has presented Figure 7, although no analytic solution exists.¹⁰

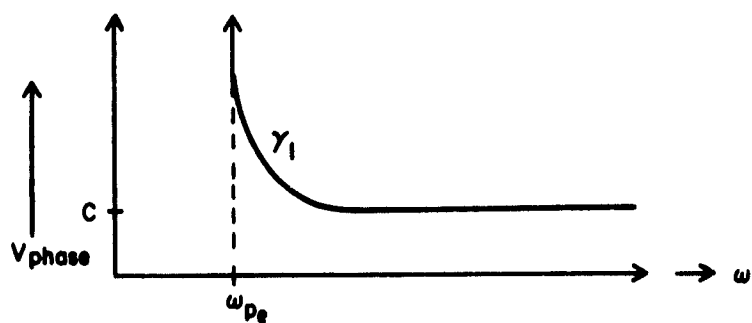
The following statements apply:

1. At low frequencies only a coupled acoustic and MHD wave can propagate. This was shown in the simplest case treated on page 20, where

$$V_{\text{phase}} = \left[\frac{2P_o k + B_o^2 / \mu}{\text{TOTAL DENSITY}} \right]$$

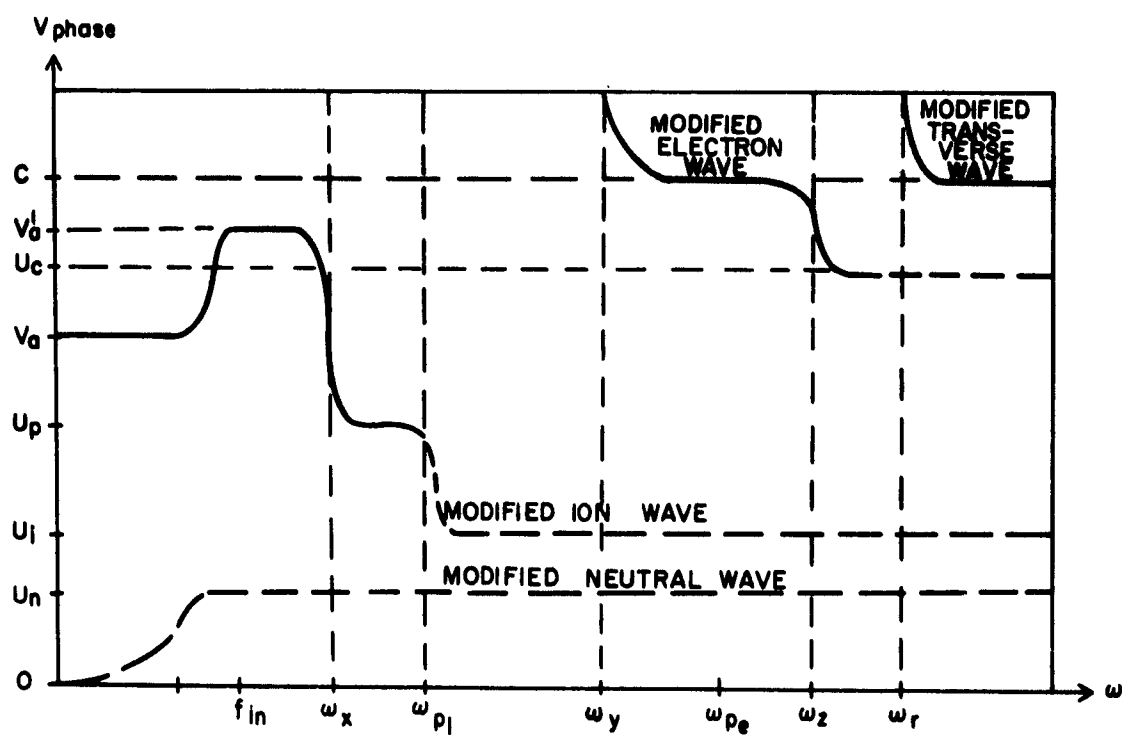
and an equivalent expression can be written for the modified-ion wave curve in this partially-ionized case as

$$V_{\text{ph}} = \left[\frac{P_k n + 2P_o k + B_o^2 / \mu}{\text{TOTAL MASS DENSITY}} \right]^{\frac{1}{2}}$$



FACTORED MODE

FIG. 6.



$$\theta = 90^\circ$$

COUPLED ACOUSTIC AND ELECTROMAGNETIC MODES

FIG. 7.

so that if the magnetic pressure is much greater than the kinetic gas pressure, the phase velocity is simply V_a , the Alfvén speed in the total gas.

2. As the frequency goes up, the neutrals are "left behind" and we have the appearance of the modified neutral wave, that has a relatively large damping factor due to neutral-ion and neutral-electron collisions, and the Alfvén mode in the charged component of gas V_a' .

3. As the frequency increases past ω_k , the effects of the magnetic pressure are lost and we have the acoustic wave in the charged component, which next becomes an acoustic wave in the ion gas alone.

4. The remaining two modes are electron oscillations - one longitudinal, the other transverse.

5. The cut-off frequencies, or transition frequencies, may vary according to the situation, but Tanenbaum has written expressions (approximate) for them.

6. These curves apply when collision effects causing damping are negligible. Tanenbaum discusses at length the effects of damping.

For an arbitrary angle, we have, of course, five modes but this situation has not been solved. It is possible to reduce the $6 \times 6 \Delta_3$ to a 4×4 determinant and obtain a quintic in γ^2 . Numerical computation then is demanded, both for the calculation of the coefficients of the polynomial and the finding of the roots. Work in this project is now being carried out, in order to develop a rapid computer program as well as enable identification of mode types.

IV. CONCLUSION

It is now clear that algebraic complications, even under the assumptions of:

1. Negligible gravity effects
2. Equal electron, ion, and neutral temperatures
3. Negligible $(\vec{V} \cdot \nabla) \vec{V}$ terms
4. Small amplitude perturbations
5. Negligible viscosity effects
6. An approximation for effective collision frequency
7. Spatial derivatives of undisturbed gas equal to zero
8. Adiabatic compressions
9. No ionization or recombination effects,

make this previously explained approach to the problem of finding a general analytic expression for the various modal dispersion relations an impossible task, and that only with numerical computation for the case of interest at the particular moment can roots be found and also that one is fortunate if the physical situation is such that certain terms can immediately be discarded. Watanabe has done an excellent paper illustrative of this approximate or intuitive method and includes tables for the important physical parameters of the ionosphere.^{5,6}

Knowledge of the exact dispersion relations will point the way to clarification of problems of propagation in the ionosphere under various conditions. Our problem has dealt with the presence of a magnetic field. If the

analysis had been performed without a field, it would have been possible to uncover the fact that the imposition of a fixed field in the vicinity of an antenna on a re-entry space vehicle should help to overcome the loss of radio contact occasioned by the hot, ionized plasma sheath.¹¹

For the last ten years, from various stations over the world, great amounts of recorded data have been obtained pertaining to micropulsations of the earth's magnetic field. Amplitudes range from fractions of a gamma ($= 10^{-5}$ gauss) to tens of gammas, with frequencies from 0.01 to 10-20 cps.¹² The main reason for studying the phenomenon is that the generation of such pulsations is thought to be related to events in outer space. Dungey in 1954 first suggested that geomagnetic micropulsations could be explained by MHD oscillations.¹³ A great amount of work has been done to develop basic equations in rectangular, cylindrical, and spherical co-ordinates to describe propagation in the slightly ionized gas of the ionosphere.^{14, 15} The effort has been to explain the observed amplitudes and change in character of the fluctuations with time of day, location, sun-spot activity, etc.¹⁶ This research is continuing.

Another problem of research interest requiring familiarity with the type of analysis of this paper is that of ionospheric heating. The attenuation factor in the propagation factor is of course pertinent here. Akasofu claims that the energy dissipation process of hydromagnetic waves in the ionosphere and the consequent heating in the F region of the ionosphere can only be negligible, of the order of 5°C under the most favorable conditions.¹⁷

The study of the propagation of signals from sudden disturbances in the troposphere, such as a lightning flash giving rise to "whistlers", or in the ionosphere, such as high-altitude bomb burst or meteoritic activity, can be clarified with the use of concepts introduced in this paper.¹⁸

REFERENCES

1. Spitzer, L., Jr., Physics of Fully Ionized Gases, Interscience Publishers, N. Y., 1956.
2. Rose, D. and M. Clark, Plasmas and Controlled Fusion, MIT Press and John Wiley and Sons, New York, 1961, Chapter 9.
3. Chapman and Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, 1953, pp. 415, 245.
4. Schluter, A., Z. Naturforschung, 5a, 72, 1950 and 69, 73, 1951.
5. Watanabe, T., Canadian Journal of Physics, 39, 1197, 1961.
6. Watanabe, T., Journal of Atmospheric and Terrestrial Physics, 24, 117, 1962.
7. Pai, S. I., Physics of Fluids, 5-2, Feb. 1962.
8. Tanenbaum, B. S., Physics of Fluids, 4, 1262 ff, (1961)
9. Ratcliffe, J. A., The Magneto-ionic Theory, Cambridge University Press, 1959.
10. Tanenbaum, B. S., Physics of Fluids, 5, 1226 ff, (1962).
11. Hodara, H., Proceedings of the IRE, 49, 1825, (1961)
12. Jacobs, J. A. and K. O. Westphal, Geophysical Journal, Roy. Astr. Soc. 6, 360-376, 1952.
13. Dungey, J. W., Penn State University Ionospheric Research Laboratory Scientific Report, No. 69, 1954.
14. Dungey, J. W., Penn State University Ionospheric Research Laboratory Scientific Report, No. 54, 1954.
15. Jacobs, J. A. and T. Watanabe, Journal Atmospheric and Terrestrial Physics, 24, 413-434, 1962.

16. Francis and Karplus, Journal of Geophysical Research, 65, 3593,
(Nov. 1960).
17. Akasofu, S., Journal of Atmospheric and Terrestrial Physics, 18,
160, (1960).
18. Northover, F. H., Journal of Atmospheric and Terrestrial Physics,
17, 158-170 (1959).

Contract Nonr 375(14)

DISTRIBUTION LIST

Addressee	Attention	No. of copies
ASTIA Arlington Hall Station Arlington 12, Virginia	TIP-Dr	10
Director of Defense Research & Development The Pentagon Washington 25, D. C.	Technical Library Room 3C-128	1
Office of the Chief Signal Officer Department of the Army Washington 25, D. C.	SIGFO-64	1
Engineering & Technical Division Office of the Chief Signal Officer Department of the Army Washington 25, D. C.	SIGET	1
Director of Research U. S. Army Signal Research & Development Laboratories Fort Monmouth, New Jersey		3
Commanding Officer U. S. Army Signal Research & Development Laboratories Fort Monmouth, New Jersey	N. Abbott	1
Administrative Engineer Signal Corps Engineering Laboratory Fort Monmouth, New Jersey		1
Chief of Naval Research Department of the Navy Washington 25, D. C.	Code 427 Code 416	2 1
Director U. S. Naval Research Laboratory Washington 25, D. C.	Code 2000 Code 2027 Code 1360	6 2 1
Chief, Bureau of Ships Department of the U. S. Navy Washington 25, D. C.	Code 337 Code 686	2 2
Chief, Bureau of Aeronautics Department of the U. S. Navy Washington 25, D. C.,	EL 50 EL 90	1 1
Department of the Navy ONR, Boston Branch Office 495 Summer St. Boston 10, Massachusetts		1

Dist. List - p. 2
Nonr 375(14)

Bureau of Naval Weapons Department of the U. S. Navy Washington 25, D. C.	RRE-3	1
Chief of Naval Operations Frequency Allocation Sect. OP-941Q Room 5E777, The Pentagon Building Washington 25, D. C.		1
Chief of Naval Operations Department of the U. S. Navy Washington 25, D. C.	Op 413	1
Director, ONR Branch Office 346 Broadway New York 13, N. Y.		1
Director, ONR Branch Office The John Crerar Library Building 10th floor 86 E. Randolph St. Chicago 1, Illinois		1
Chief Scientist ONR Branch Office 1030 Green Street (East) Pasadena 1, California		1
Officer in Charge, ONR Navy No. 100 - Fleet Post Office New York, N. Y.		1
Commanding Officer U. S. Naval Underwater Sound Laboratory New London, Connecticut		1
Commanding Officer Naval Air Development Center Johnsville, Pennsylvania	AAEL	1
Director Naval Ordnance Laboratory White Oak, Maryland		1
Commander U. S. Naval Ordnance Test Station China Lake, California	Technical Library Code 753	1
Librarian, U. S. Naval Postgraduate School Monterey, California		1
U. S. Naval Electronics Laboratory San Diego 52, California	Library	1

Dist. List - p. 3
Nonr 375(14)

Alderman Library University of Virginia Charlottesville, Virginia	John C. Wylie	1
Commanding General Hq., USAF Washington 25, D. C.	AC-AS/4:AFDRE	1
Aeronautical Systems Division Deputy for Technology-Avionics Div. Electromagnetic Warfare & Comm- ications Electromagnetic Environment Branch- Propagation Section Wright-Patterson AFB, Ohio	Paul W. Springer ASRNCF-2	1
AF Cambridge Research Laboratories	CRRK	1
Electronics Research Directorate	CRRI	1
Office of Aerospace Research	CRRD	1
L. G. Hanscom Field Bedford, Massachusetts	CRRS	1
AF Cambridge Research Laboratories Geophysics Research Directorate Office of Aerospace Research L. G. Hanscom Field Bedford, Massachusetts		1
Commander AF Office of Scientific Research Washington, D. C.	Dr. Wm. J. Otting, Jr.	1
U. S. Weather Bureau Department of Commerce Washington, D. C.	Director Radar Engineering	1
Boulder Laboratories National Bureau of Standards Boulder, Colorado	Library W. H. Campbell Ionospheric Res. & Propagation Div.	1 1
Office of Technical Services U. S. Department of Commerce Washington 25, D. C.		1
ITT Laboratories 3700 E. Pontiac Street Fort Wayne, Indiana	Mrs. Ann Bley	1
Maurice W. Long Engineering Experiment Station Georgia Institute of Technology Atlanta 13, Georgia		1

Dist. List - p. 4
Nonr 375(14)

Department of EE University of Michigan Ann Arbor, Michigan	Dr. S. S. Atwood	1
Research Laboratory of Electronics M. I. T. Cambridge, Massachusetts	H. Zimmerman	1
Federal Communications Commission Room 2209, New P. O. Bldg. Washington 25, D. C.	Asst. Chief Engineer in charge of Technical Research Division	1
Stanford Electronics Laboratories Stanford University Stanford, California	Applied Electronics Lab Document Library	1
Library, ERL 427 Corey Hall, Univ. of California Berkeley 4, California	Miss L. Khouri	1
Documents Library R-3446 General Electric Company Missile and Space Vehicle Dept. 3198 Chestnut Street Philadelphia 4, Pa.	Lawrence I. Chasen Mgr., Library	1
Bell Telephone Company P. O. Box 107 Red Bank, New Jersey	Dr. R. Kompfner	1
Dr. William A. Miller Senior Staff Scientist Electronics Systems Division Fairchild Stratat Corp. Straight Path Wyandanch, N. Y.		1
Electrical Engineering Res. Labs. University of Illinois Urbana, Illinois	P. D. Coleman	1
RCA Laboratories Princeton, New Jersey	Miss Fern Cloak	1
Technical Reports Collection Gordon McKay Library Harvard University Div. of Eng'g. and Appl. Physics Pierce Hall, Oxford Street Cambridge 38, Massachusetts	Mrs. Elizabeth Farkas	1
School of Electrical Engineering Cornell University Ithaca, New York	Dr. W. E. Gordon	1

Dist. List - p. 5
Nonr 375(14)

Commandant (EEE) U. S. Coast Guard (Sta 5-5) 1300 E Street, N. W. Washington 25, D. C.		1
M. I. T., Lincoln Laboratory Box 73 Lexington 73, Massachusetts	Dr. James H. Chisholm	1
Dept. of Electrical Engineering California Institute of Technology Pasadena, California	Dr. C. H. Papas	1
Applied Physics Laboratory The Johns Hopkins University 8621 Georgia Avenue Silver Spring, Maryland	Geo. L. Seielstad	2
New York University Institute of Mathematical Sciences 25 Waverly Place New York 3, N. Y.	Prof. Morris Kline	1
Program Director, Eng'g. Sciences National Science Foundation Washington 25, D. C.		1
Antenna Research Laboratory Ohio State University Columbus 10, Ohio	Dr. Thomas Tice	1
Tektronix, Inc. P. O. Box 500 Beaverton, Oregon	Library	1
Smyth Research Associates 3555 Aero Court San Diego 11, California	Electronics Branch	1
Cornell Aeronautical Laboratory, Inc. 4455 Genessee Street Buffalo 21, New York	John P. Desmond, Librarian	1
New York University College of Engineering Dept. of Meteorology & Oceanography University Heights New York 53, N. Y.	Prof. W. J. Pierson	1
Dr. Nicholas George 333 Spalding Building California Institute of Technology Pasadena, California		1

Dist. List - p. 6
Nonr 375(14)

Texas A & M College College Station, Texas	Dr. Vance Moyer, Radar Met. Section	3
Sylvania Electric Products, Inc. P. O. Box 205 Mountain View, California	Miss Katherine Johnson Electronics Defense Lab.	1
Radio Research Laboratories Ministry of Post & Telecommunications Kokubunji Post Office Tokyo, Japan VIA: U. S. Naval Attache for Air Tokyo, Japan	Planning Section	1
Harvard Observatory 60 Garden Street Cambridge 38, Massachusetts	A. E. Lilley Dept. of Radio Astronomy	1
University of Arizona Tucson, Arizona	Mr. George Savage Inst. of Atmos. Physics	1
Technical Research Group, Inc. 2 Aerial Way Syosset, New York	Jerome R. Lurye	1
Loral Electronics Corporation 835 Bronx River Avenue New York 72, N. Y.	W. Honig	1
National Bureau of Standards P. O. Box 299 Boulder, Colorado	Mr. M. C. Thompson	1
Research Laboratory of Electronics Chalmers Institute of Technology Gothenburg, Sweden	Christina Walsh	1
Advanced Research Projects Agency Room 3D 170, The Pentagon Washington 25, D. C.	Lt. Cmdr. D. E. Chandler	1
Alan H. Barrett, Assoc. Prof. Room 26-49 M. I. T. Cambridge 39, Massachusetts		1
Hughes Aircraft Company Communications Division Box 90902 Los Angeles, California		1

Dist. List - p. 7
Nonr 375(14)

Dr. C. I. Beard Boeing Scientific Research Labs. P. O. Box 3981 Seattle 24, Washington		1
Mr. Charles R. Burrows Senior Scientist Radio Eng'g. Labs., Inc. 4925th Street, Elmo Ave. Bethesda 14, Maryland		1
University of Chicago Laboratories for Applied Sciences Museum of Science & Industry Chicago 37, Illinois		1
Instituto Geofisico del Peru Apartado 3747 Lima, Peru	Dr. Robert Cohen	1
Jet Propulsion Laboratory California Institute of Technology 4800 Oak Grove Drive Pasadena, California	I. E. Newlan Technical Reports Section	1
Manila Observatory Ateneo De Manila University P. O. Box 1231 Manila, P. I.		1
Geophysical and Polar Research Center University of Wisconsin 6021 South Highland Road The Highlands Madison 5, Wisconsin	Mr. Forrest L. Dowling	1
Director U. S. Army Engineer Research and Development Laboratories Fort Belvoir, Virginia	Technical Documents Center	1
U.S. Army Materiel Command Harry Diamond Laboratories Room 211, Building 92 Washington 25, D. C.	Mrs. A. Porter	1